

# ON THE EIGENCURVE AT CLASSICAL WEIGHT ONE POINTS

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ABSTRACT. We show that the  $p$ -adic Eigencurve is smooth at classical weight one points which are regular at  $p$  and give a precise criterion for etaleness over the weight space at those points. Our approach uses deformations of Galois representations.

## 1. INTRODUCTION

A well known result of Hida [13] asserts that the Eigencurve is etale over the weight space at all classical ordinary points of weight two or more, hence is smooth at those points. This result has been generalized to all non-critical classical points of weight two or more by Coleman and Mazur [9], when the tame level is one, and by Kisin [14] for arbitrary tame level.

The aim of this paper is to further investigate the geometry of the Eigencurve at classical points of weight one. Note that these points are both ordinary and critical.

In order to state our results, we need to fix some notations. We let  $\bar{\mathbb{Q}} \subset \mathbb{C}$  be the field of algebraic numbers and denote by  $G_L$  the absolute Galois group of a perfect field  $L$ . When  $L$  is a number field and  $M$  a positive integer, we denote by  $G_{L,M}$  the Galois group of the maximal extension of  $L$  unramified except at finite places dividing  $M$  and at infinite places, and for a prime  $\ell$  that does not divide  $M$ , by  $\text{Frob}_\ell$  the arithmetic Frobenius element at  $\ell$  in  $G_{L,M}$  which is only defined up to conjugacy.

Let  $f(z) = \sum_{n \geq 1} a_n e^{2\sqrt{-1}\pi n z}$  be a newform of weight one, level  $M$  and nebentypus  $\epsilon$ . It is a theorem of Deligne and Serre [10] that there exists a continuous irreducible representation with finite image:

$$\rho_f : G_{\mathbb{Q},M} \rightarrow \text{GL}_2(\mathbb{C}),$$

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such that  $\text{Tr } \rho_f(\text{Frob}_\ell) = a_\ell$  and  $\det \rho_f(\text{Frob}_\ell) = \epsilon(\ell)$  for all  $\ell \nmid M$ . For  $\ell$  a prime, let  $\alpha_\ell$  and  $\beta_\ell$  be the roots (possibly equal) of the Hecke polynomial  $X^2 - a_\ell X + \epsilon(\ell)$ . When  $\ell \nmid M$ , the Hecke polynomial is the characteristic polynomial of  $\rho_f(\text{Frob}_\ell)$ . We say that  $f$  is *regular* at  $\ell$  if  $\alpha_\ell \neq \beta_\ell$ .

During all this article, we fix a prime number  $p$  and an embedding  $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ . Since the representation  $\rho_f$  has finite image, it is equivalent to a representation whose image is in  $\text{GL}_2(\bar{\mathbb{Q}})$  and using the embedding  $\iota_p$  we can see  $\rho_f$  as a representation  $G_{\mathbb{Q},M} \rightarrow \text{GL}_2(\bar{\mathbb{Q}}_p)$ . Similarly  $\alpha_p$  and  $\beta_p$  are algebraic and we see them as elements of  $\bar{\mathbb{Q}}_p$  using  $\iota_p$ .

In order to deform  $p$ -adically  $f$ , one must first choose a  $p$ -*stabilization* of  $f$  with finite slope, that is an eigenform of level  $\Gamma_1(M) \cap \Gamma_0(p)$  sharing the same eigenvalues as  $f$  away from  $p$  and having a non-zero  $U_p$ -eigenvalue. This is done differently according to whether  $p$  divides the level or not. If  $p \nmid M$ , we define  $f_\alpha(z) = f(z) - \beta_p f(pz)$  and  $f_\beta(z) = f(z) - \alpha_p f(pz)$ . Those forms are  $p$ -stabilizations of  $f$  with  $U_p$ -eigenvalues  $\alpha_p$  and  $\beta_p$ , respectively. Note that  $f_\alpha$  and  $f_\beta$  are distinct if, and only if,  $f$  is regular at  $p$ . If  $p$  divides  $M$  and  $a_p \neq 0$  (that is if  $f$  is regular at  $p$ ), then  $U_p \cdot f = a_p f$  and we simply set  $\alpha_p = a_p$  and  $f_\alpha = f$ . It follows from [10, Theorem 4.2] that the  $U_p$ -eigenvalue of any  $p$ -stabilization  $f_\alpha$  is a root of unity and *a fortiori* a  $p$ -adic unit. Hence, if  $f$  is regular at  $p$ , then  $f_\alpha$  is ordinary at  $p$ .

Denote by  $N$  the prime to  $p$ -part of  $M$ . Let  $\mathcal{C}$  be the Eigencurve of tame level  $N$ , constructed with the Hecke operators  $U_p$  and  $T_\ell$  for  $\ell \nmid Np$ . There exists a flat and locally finite  $\kappa : \mathcal{C} \rightarrow \mathcal{W}$ , where the weight space  $\mathcal{W}$  is the rigid analytic space over  $\mathbb{Q}_p$  representing continuous group homomorphisms from  $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$  to  $\mathbb{G}_m$ , and  $f_\alpha$  defines a point on  $\mathcal{C}$ , whose image by  $\kappa$  has finite order (see §6 for more details).

**Theorem 1.1.** *Let  $f$  be a classical weight one newform of level  $\Gamma_1(M)$  which is regular at  $p$ . Then the Eigencurve  $\mathcal{C}$  is smooth at  $f_\alpha$ . Moreover  $\kappa$  is etale at  $f_\alpha$  unless there exists a real quadratic field  $K$  in which  $p$  splits and such that  $\rho_{f|_{G_K}}$  is reducible.*

The latter case was first investigated by Cho and Vatsal in [8] where they show, under some additional assumptions, that the ramification index is exactly 2. We have recently learned that Greenberg and Vatsal have announced a result similar to ours under the assumption that the adjoint representation is regular at  $p$ . If  $f$  is exceptional and regular at  $p$ , the theorem implies that  $\kappa$  is etale at  $f_\alpha$ . For

example, the smallest irregular prime for the exceptional weight one newform of level 133 found by Tate is 163.

Let us observe already that there is alternative construction of the Eigencurve which uses also the operators  $U_\ell$  for  $\ell \mid N$ . However, had we decided to work with this Eigencurve, by corollary 8.3, the results would have been exactly the same.

**Corollary 1.2.** *Let  $f$  be a classical weight one newform of level  $\Gamma_1(M)$ , which is regular at  $p$ . Then there is a unique irreducible component of  $\mathcal{C}$  containing  $f_\alpha$ . If  $f$  has CM by a quadratic imaginary field  $K$  in which  $p$  is split, then all classical points of that component also have CM by  $K$ .*

Our interest in studying the geometry of the Eigencurve at classical weight one points arose from questions about specializations of primitive Hida families in weight one. In that language the above corollary says that there exists a unique Hida family specializing to a given  $p$ -stabilized, regular, classical weight one newform. If one relaxes the condition of regularity, then there exist distinct primitive Hida families specializing to the same  $p$ -stabilized classical weight one newform (see [11]). Studying the geometry of the Eigencurve at the corresponding point, not only could answer the question of how many families specialize to that eigenform, but could also potentially give a deeper insight on how exactly those families meet.

Another motivation is the following application to the theory of  $p$ -adic  $L$ -functions. Using some ideas of Mazur, Kitagawa has constructed more than twenty years ago a  $p$ -adic  $L$ -functions for Hida's families, under certain technical assumptions (see conditions (A) or (B) in [15, p.105]). Their main property is that any specialization in weight at least two equals, up to a  $p$ -adic unit, the  $p$ -adic  $L$ -function  $L_p(g, s)$  attached to a  $p$ -stabilized ordinary eigenform  $g$  of that weight by Mazur, Manin, Vishik, Amice-Velu *et al.* (see [16]). The smoothness established in theorem 1.1 allows us, by a method used in [2] and detailed in [3], to give a weaker but unconditional version of the Mazur-Kitagawa construction. Denote  $\mathcal{W}^+$  (resp.  $\mathcal{W}^-$ ) the subspace of  $\mathcal{W}$  consisting of homomorphisms sending  $-1$  to  $1$  (resp. to  $-1$ ). By [3, Corollary VI.4.2], we obtain:

**Corollary 1.3.** *Let  $f$  be a classical weight one newform of level  $\Gamma_1(M)$  which is regular at  $p$ . Then for every sufficiently small admissible neighborhood  $U$  of  $f_\alpha$  in  $\mathcal{C}$ , there exist two real constants  $0 < c < C$ , and two two-variables analytic functions  $L_p^\pm$  on  $U \times \mathcal{W}^\pm$ , such that for every eigenform  $g \in U$  of weight at least*

two, and every  $s \in \mathcal{W}^\pm$

$$L_p^\pm(g, s) = e^\pm(g) L_p(g, s),$$

where  $e^\pm(g)$  is a  $p$ -adic period such that  $c < |e^\pm(g)|_p < C$ . Moreover for every  $g \in U$  the functions  $s \mapsto L_p^\pm(g, s)$  are bounded.

As explained in *loc. cit.*, it is easy to see that while the two functions  $L_p^\pm$  are not uniquely determined by the conditions of the corollary, each of the functions  $L_p^\pm(f_\alpha, s)$  on  $\mathcal{W}^\pm$  is uniquely determined up to a non-zero multiplicative constant. The resulting function on  $\mathcal{W}$  is thus well-defined up to two multiplicative non-zero constants, one on  $\mathcal{W}^+$  and one on  $\mathcal{W}^-$ , and seems to be a natural definition for the analytic  $p$ -adic  $L$ -function  $L_p(f_\alpha, s)$  of  $f_\alpha$ . The zeros of  $L_p(f_\alpha, s)$  with their multiplicities are well-defined and are finitely many by the boundedness property.

Let us now explain the main ideas behind the proof of Theorem 1.1. In §2 we introduce a deformation problem for  $\rho_f$  representable by a  $\bar{\mathbb{Q}}_p$ -algebra  $\mathcal{R}$  which by §7 surjects to the completed local ring of the Eigencurve at  $f_\alpha$ . The computation of the tangent space of  $\mathcal{R}$  represents an important part of the proof and shows that its dimension is always 1, hence the above surjection is an isomorphism and the Eigencurve is smooth at  $f_\alpha$  (the question of the etaleness is treated parallelly by the studying the algebra of the fiber of the weight map at  $f_\alpha$ ). The tangent space is interpreted as usual as a subspace of  $H^1(\mathbb{Q}, \text{ad } \rho_f)$  satisfying certain local condition, but since  $\text{ad } \rho_f$  has finite image, it can be further viewed as a subspace of  $\text{Hom}(G_H, \bar{\mathbb{Q}}_p)^4$ , for some finite Galois extension  $H$  of  $\mathbb{Q}$ . Its dimension can be easily deduced from a general version of the Leopoldt conjecture for  $H$ , which is not available for all the  $H$  we have to consider. To circumvent this problems, we use the Baker-Brumer theorem (the very theorem of transcendence theory used to prove Leopoldt's conjecture for all abelian extension of  $\mathbb{Q}$  or of an imaginary quadratic field) together with a crucial lemma saying that our tangent space has a basis of cocycles defined over  $\bar{\mathbb{Q}}$ . The setting for this computation is developed in §2 and §3 while the computation itself is performed in §4 and §5.

## 2. A DEFORMATION PROBLEM

For every place  $v$  of a number field  $L$  above a prime number  $\ell$ , by choosing an appropriate embedding of  $\bar{\mathbb{Q}}$  in  $\bar{\mathbb{Q}}_\ell$ , one can see  $G_{L_v}$  as a decomposition subgroup of  $G_L$  at  $v$ . We denote  $I_v$  the inertia subgroup of  $G_{L_v}$ .

Let  $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$  be an irreducible  $p$ -adic representation with *finite image*. We say that  $\rho$  is odd (resp. even) if  $\det \rho(\tau) = -1$  (resp. 1), where  $\tau$  is the complex conjugation.

We assume that  $\rho$  is *ordinary* at  $p$ , meaning that the restriction of  $\rho$  to  $G_{\mathbb{Q}_p}$  is a sum of two characters  $\psi_1$  and  $\psi_2$ , with  $\psi_1$  unramified, and choose a basis  $\{e_1, e_2\}$  of  $\bar{\mathbb{Q}}_p^2$  in which  $\rho|_{G_{\mathbb{Q}_p}} = \psi_2 \oplus \psi_1$ .

Assume also that  $\rho$  is *regular* at  $p$ , that is  $\psi_1 \neq \psi_2$ . If both characters  $\psi_1$  and  $\psi_2$  are unramified, we will privilege  $\psi_1$ , so that the basis  $\{e_1, e_2\}$  is unique up to scaling.

We consider the following deformation problem attached to  $(\rho, \psi_1)$ : for  $A$  any local Artinian ring with maximal ideal  $\mathfrak{m}_A$  and residue field  $A/\mathfrak{m}_A = \bar{\mathbb{Q}}_p$ , we define  $\mathcal{D}(A)$  as the set of strict equivalence classes of representations  $\rho_A : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(A)$  lifting  $\rho$  (that is  $\rho_A \bmod \mathfrak{m}_A \simeq \rho$ ) and which are ordinary at  $p$  in the sense that:

$$(1) \quad \rho_A|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \psi_{2,A} & * \\ 0 & \psi_{1,A} \end{pmatrix},$$

where  $\psi_{1,A} : G_{\mathbb{Q}_p} \rightarrow A^\times$  is an unramified character lifting  $\psi_1$ .

Let  $\mathcal{D}'$  be the subfunctor of  $\mathcal{D}$  of deformation with constant determinant. We call  $t_{\mathcal{D}'}$  and  $t_{\mathcal{D}}$  the tangent spaces to those functors.

Since  $\rho|_{G_{\mathbb{Q}_p}} = \psi_2 \oplus \psi_1$ , we have a natural decomposition

$$(\mathrm{ad} \rho)|_{G_{\mathbb{Q}_p}} = \psi_2/\psi_2 \oplus \psi_1/\psi_2 \oplus \psi_2/\psi_1 \oplus \psi_1/\psi_1.$$

Therefore we have natural  $G_{\mathbb{Q}_p}$ -equivariant maps:  $\mathrm{ad} \rho \rightarrow \psi_2/\psi_1$  and  $\mathrm{ad} \rho \rightarrow \psi_1/\psi_1$ , which we use implicitly in the following standard lemma in deformation theory.

**Lemma 2.1.**

$$(2) \quad t_{\mathcal{D}} = \ker \left( H^1(\mathbb{Q}, \mathrm{ad} \rho) \rightarrow H^1(\mathbb{Q}_p, \psi_2/\psi_1) \oplus H^1(I_p, \psi_1/\psi_1) \right)$$

$$(3) \quad t_{\mathcal{D}'} = \ker \left( H^1(\mathbb{Q}, \mathrm{ad}^0 \rho) \rightarrow H^1(\mathbb{Q}_p, \psi_2/\psi_1) \oplus H^1(I_p, \psi_1/\psi_1) \right)$$

The following lemma will be used to transform the above expression of  $t_{\mathcal{D}}$  under inflation-restriction sequences.

**Lemma 2.2.** *Let  $L$  be a number field, and  $\rho$  a representation of  $G_L$ .*

- (i) *Let  $\Sigma$  be a set of places of  $L$ , and for every  $v \in \Sigma$ , let  $\rho_v$  be a quotient of  $\rho|_{G_{L_v}}$ . Let  $H$  be a Galois extension of  $L$  and for all  $v \in \Sigma$ , let  $w(v)$  be a*

place of  $H$  above  $v$ . Then

$$\ker \left( H^1(L, \rho) \rightarrow \prod_{v \in \Sigma} H^1(L_v, \rho_v) \right) \simeq \ker \left( H^1(H, \rho)^{\text{Gal}(H/L)} \rightarrow \prod_{v \in \Sigma} H^1(H_{w(v)}, \rho_v) \right).$$

(ii) Assume that  $\rho$  is a  $p$ -adic representation with finite image. Then the natural homomorphism  $H^1(L, \rho) \rightarrow \prod_{v|p} H^1(I_v, \rho)$  is injective.

Denote by  $H \subset \bar{\mathbb{Q}}$  the fixed field of  $\ker(\text{ad } \rho)$ . The group  $G = \text{Gal}(H/\mathbb{Q})$  is naturally isomorphic to the projective image of  $\rho$ , which is known to be either dihedral or exceptional ( $\simeq A_4, S_4$  or  $A_5$ ).

Using the basis  $\{e_1, e_2\}$  defined above one can see an element of  $H^1(\mathbb{Q}, \text{ad } \rho)$  as a cocycle

$$G_{\mathbb{Q}} \rightarrow M_2(\bar{\mathbb{Q}}_p), \quad g \mapsto \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}.$$

By the inflation-restriction sequence, we have an isomorphism:

$$H^1(\mathbb{Q}, \text{ad } \rho) \simeq (\text{Hom}(G_H, \bar{\mathbb{Q}}_p) \otimes \text{ad } \rho)^G,$$

which enables us to see  $a, b, c, d$  as elements of  $\text{Hom}(G_H, \bar{\mathbb{Q}}_p)$ .

By lemma 2.2(ii) applied to the trivial representation and by class field theory we have a natural injective homomorphism

$$(4) \quad H^1(H, \bar{\mathbb{Q}}_p) \hookrightarrow \text{Hom}((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, \bar{\mathbb{Q}}_p),$$

hence one can further see  $a, b, c$  and  $d$  as elements of  $\text{Hom}((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, \bar{\mathbb{Q}}_p)$ . The image of (4) consists precisely of those homomorphism which are trivial on the subgroup  $\mathcal{O}_H^\times$  of global units. In other terms:

$$H^1(H, \bar{\mathbb{Q}}_p) \simeq \text{Hom}((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times / \bar{\mathcal{O}}_H^\times, \bar{\mathbb{Q}}_p),$$

where the closure  $\bar{\mathcal{O}}_H^\times$  of  $\mathcal{O}_H^\times$  is taken with respect to the  $p$ -adic topology.

One has  $\mathcal{O}_H \otimes \mathbb{Z}_p \simeq \prod_{w|p} \mathcal{O}_{H,w}$ , where  $w$  runs over all places of  $H$  above  $p$  and  $\mathcal{O}_{H,w}$  is the ring of integers of the completion  $H_w$ .

A continuous homomorphism  $\text{Hom}(\mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p)$  is of the form

$$(5) \quad u \mapsto \sum_{g_w \in J_w} h_{g_w} g_w(\log_p(u)) = \sum_{g_w \in J_w} h_{g_w} \log_p(g_w(u)),$$

for some  $h_{g_w} \in \bar{\mathbb{Q}}_p$ , where  $J_w$  is the set of all embeddings of  $H_w$  in  $\bar{\mathbb{Q}}_p$  and  $\log_p$  is the standard determination of the  $p$ -adic logarithm on  $\bar{\mathbb{Q}}_p^\times$  which is trivial on roots of unity of order prime to  $p$  (alternatively, one can restrict to a finite index subgroup of  $\mathcal{O}_{H,w}^\times$  where the  $p$ -adic logarithm is defined by the usual power series).

The embedding  $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$  and the inclusions  $H \subset \bar{\mathbb{Q}} \subset \mathbb{C}$  define a partition

$$G = \coprod_{w|p} J_w$$

coming from the following commutative diagram:

$$\begin{array}{ccc} \bar{\mathbb{Q}} & \xrightarrow{\iota_p} & \bar{\mathbb{Q}}_p \\ g \uparrow & & \uparrow g_w \\ H & \hookrightarrow & H_w. \end{array}$$

It follows that there is a canonical isomorphism:

$$(6) \quad \begin{aligned} & \bar{\mathbb{Q}}_p[G] \xrightarrow{\sim} \text{Hom}((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, \bar{\mathbb{Q}}_p) \\ & \sum_{g \in G} h_g g \mapsto \left( u \otimes 1 \mapsto \sum_{g \in G} h_g \log_p(\iota_p \circ g(u)) \right). \end{aligned}$$

We denote by  $\sum_{g \in G} a_g g$  the inverse image of  $a$  in  $\bar{\mathbb{Q}}_p[G]$ , and similarly for  $b, c$  and  $d$ .

We will now bound the multiplicities  $m_\pi \in \mathbb{Z}$  occurring in the decomposition

$$(7) \quad H^1(H, \bar{\mathbb{Q}}_p) \simeq \bigoplus_{\pi} \pi^{m_\pi},$$

where  $\pi$  runs over all characters of  $G$ . By (6),  $\text{Hom}((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, \bar{\mathbb{Q}}_p)$  is isomorphic to the regular representation which can be decomposed as  $\bigoplus_{\pi} \pi^{\dim \pi}$ . By Minkowski's proof of Dirichlet's unit theorem, the  $G$ -module  $\mathcal{O}_H^\times \otimes \bar{\mathbb{Q}}_p$  is isomorphic to  $\bigoplus_{\pi \neq 1} \pi^{\dim \pi^+}$ , where  $\pi^+$  and  $\pi^-$  are the eigenspaces for the action of the complex conjugation  $\tau \in G$ .

**Lemma 2.3.** *We have  $m_1 = 1$  and for  $\pi \neq 1$  we have  $\dim \pi^- \leq m_\pi \leq \dim \pi$  with equality  $m_\pi = \dim \pi^-$  if Leopoldt's conjecture holds for  $H$ .*

Contrarily to what the previous lemma suggests, it turns out that Leopoldt's conjecture for  $H$  is not needed in order to compute  $t_{\mathcal{D}'}$ . A crucial observation is that the latter has a basis defined over the algebraic numbers.

**Lemma 2.4.** *Assume that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in t_{\mathcal{D}'} \subset H^1(\mathbb{Q}, \text{ad}^0 \rho)$  with  $b_1 \in \bar{\mathbb{Q}}$ . Then for all  $g \in G$  we have  $a_g, b_g, c_g, d_g \in \bar{\mathbb{Q}}$  and  $t_{\mathcal{D}'}$  has dimension at most 1.*

*Proof.* The embedding  $\iota_p$  determines a place  $w_0$  of  $H$  above  $p$  and allows us to see  $G_{H_{w_0}}$  as a decomposition subgroup of  $G_H$  at  $w_0$ . By Lemmas 2.1 and 2.2(i) an element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in t_{\mathcal{D}'}$  belongs to the kernel of the homomorphism

$$(H^1(H, \bar{\mathbb{Q}}_p) \otimes \text{ad}^0 \rho)^{\text{Gal}(H/K)} \rightarrow H^1(H_{w_0}, \bar{\mathbb{Q}}_p) \oplus H^1(I_{w_0}, \bar{\mathbb{Q}}_p),$$

hence  $c_1 = 0$  and  $a_1 = -d_1 = 0$ . Since  $b_1 \in \bar{\mathbb{Q}}$  and  $\rho(g) \in \text{GL}_2(\bar{\mathbb{Q}})$ , for all  $g \in G$ , the lemma follows from the cocycle relation

$$(8) \quad \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} = \rho(g)^{-1} \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix} \rho(g).$$

□

### 3. BAKER-BRUMER'S THEOREM

Let  $n$  denote the number of infinite places of  $H$ . If  $\rho$  is even, then the order of  $G$  is  $n$ , whereas if  $\rho$  is odd, then the order of  $G$  is  $2n$ . If  $\rho$  is odd the inclusion  $H \subset \mathbb{C}$  yields a complex conjugation  $\tau \in G$ . Let  $\{g_1, \dots, g_n\}$  be a set of representatives of  $\{1, \tau\} \backslash G$ . By Minkowski's proof of Dirichlet's unit theorem there exists  $u_0 \in \mathcal{O}_H^\times$  such that  $\{g_1(u_0), \dots, g_{n-1}(u_0)\}$  is a basis of  $\mathcal{O}_H^\times \otimes \mathbb{Q}$ . Equivalently, one can formulate this by saying that the image of the  $\mathbb{Q}$ -linear homomorphism

$$\Psi : \mathbb{Q}[G] \rightarrow \bar{\mathbb{Q}}_p, \quad \sum_{g \in G} h_g g \mapsto \sum_{g \in G} h_g \log_p (\iota_p \circ g(u_0))$$

has  $\mathbb{Q}$ -rank  $n - 1$ . It is easy to check that  $\ker(\Psi) \subset \mathbb{Q}[G]$  is a left ideal and that the above homomorphism induces a  $G$ -equivariant isomorphism  $\mathbb{Q}[G]/\ker(\Psi) \simeq \mathcal{O}_H^\times \otimes \mathbb{Q}$ . Surprisingly a similar statement continue to hold over  $\bar{\mathbb{Q}}$ .

**Theorem 3.1.** (*Baker-Brumer [4]*) *The image of the  $\bar{\mathbb{Q}}$ -linear homomorphism*

$$(9) \quad \bar{\Psi} : \bar{\mathbb{Q}}[G] \rightarrow \bar{\mathbb{Q}}_p, \quad \sum_{g \in G} h_g g \mapsto \sum_{g \in G} h_g \log_p (\iota_p \circ g(u_0))$$

*has  $\bar{\mathbb{Q}}$ -rank  $n - 1$ .*

Let us record the following useful consequence:

**Corollary 3.2.** (i)  $\ker(\bar{\Psi}) \subset \bar{\mathbb{Q}}[G]$  *is a left ideal and the homomorphism (9) induces a  $G$ -equivariant isomorphism*

$$\bar{\mathbb{Q}}[G]/\ker(\bar{\Psi}) \simeq \mathcal{O}_H^\times \otimes \bar{\mathbb{Q}}.$$



- (ii) Let  $h \in \text{Hom}(G_H, \bar{\mathbb{Q}}_p)$  be such that the element  $\sum_{g \in G} h_g g \in \bar{\mathbb{Q}}_p[G]$  (associated to  $h$  using (4) and (6)) belongs to  $\bar{\mathbb{Q}}[G]$ . Then the bilateral ideal generated by  $\sum_{g \in G} h_g g$  is contained in  $\ker(\bar{\Psi})$ .

*Proof.* Theorem 3.1 yields  $\ker(\bar{\Psi}) \simeq \ker(\Psi) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$  which implies (i).

For the right invariance by  $G$  in (ii), note that since  $h$  is trivial on global units, in particular on those of the form  $g'(u_0) \otimes 1$  with  $g' \in G$ , one has  $\sum_{g \in G} h_g g g' \in \ker(\bar{\Psi})$  for all  $g' \in G$ .  $\square$

#### 4. THE EVEN AND EXCEPTIONAL CASES

Keep the assumptions on  $\rho$  from §2.

**Proposition 4.1.** *If  $\rho$  is even, then  $\ker(\bar{\Psi}) = \bar{\mathbb{Q}}(\sum_{g \in G} g)$  and  $t_{\mathcal{D}'} = 0$ .*

*Proof.* Clearly  $\ker(\bar{\Psi}) \supset \bar{\mathbb{Q}}(\sum_{g \in G} g)$  and theorem 3.1 implies the equality by comparing the dimensions.

By lemma 2.4 if  $t_{\mathcal{D}'} \neq 0$ , then it has a basis  $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in t_{\mathcal{D}'}$  with  $a_g, b_g, c_g \in \bar{\mathbb{Q}}$  for all  $g \in G$ . It follows then from corollary 3.2(ii) that  $\sum_{g \in G} a_g g$ ,  $\sum_{g \in G} b_g g$  and  $\sum_{g \in G} c_g g$  belong to  $\ker(\bar{\Psi})$ . Since  $a_1 = c_1 = 0$ , it follows that  $a = c = 0$  and the cocycle relation (8) then implies that  $\rho$  is reducible which is a contradiction.  $\square$

**Theorem 4.2.** *Assume that  $\rho$  is odd and that  $G \simeq A_4, S_4$  or  $A_5$ . Then  $t_{\mathcal{D}'} = 0$  and  $t_{\mathcal{D}}$  has dimension at most 1.*

*Proof.* Let  $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in t_{\mathcal{D}'}$ . As in the proof of corollary 4.1, one can assume that  $\sum_{g \in G} a_g g$ ,  $\sum_{g \in G} b_g g$  and  $\sum_{g \in G} c_g g$  belong to  $\ker(\bar{\Psi})$  and  $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

By corollary 3.2(i) and the discussion following (7) we have:

$$\bar{\mathbb{Q}}[G]/\ker(\bar{\Psi}) \simeq \mathcal{O}_H^\times \otimes \bar{\mathbb{Q}} \simeq \bigoplus_{\pi \neq 1} \pi^{\dim \pi^+}.$$

Using the character tables of  $A_4, S_4$  and  $A_5$ , one finds that for every irreducible representation  $\pi$  of  $G$  one has  $\dim \pi^+ \neq 0$ , except possibly for the sign representation  $\varepsilon$  of  $S_4$ . This allows us to compute the maximal bilateral ideal  $J$  of  $\bar{\mathbb{Q}}[G]$  contained in  $\ker(\bar{\Psi})$ :

$\tau \in G$	$(12)(34) \in A_4$				$(12)(34) \in A_5$					$(12)(34) \in S_4$					$(12) \in S_4$				
$\dim \pi$	1	1	1	3	1	3	3	4	5	1	1	2	3	3	1	1	2	3	3
$\dim \pi^+$	1	1	1	1	1	1	1	2	3	1	1	2	1	1	1	0	1	1	2
$J$	$(\sum_g g)$				$(\sum_g g)$					$(\sum_g g)$					$(\sum_g g, \sum_g \varepsilon(g)g)$				

In the first three cases,  $a_1 = c_1 = d_1 = 0$  implies that  $a = c = d = 0$ , hence  $\rho$  is reducible.

In the last case ( $G \simeq S_4$  and  $\tau = (12)$ ) one finds that  $\sum_{g \in G} a_g g$ ,  $\sum_{g \in G} (b_g - 1)g$  and  $\sum_{g \in G} c_g g$  belong to  $\bar{\mathbb{Q}}(\sum_g (1 - \varepsilon(g))g)$ . Let  $\phi \in G$  be a non-trivial element of the decomposition group at  $w_0$  (it exists by  $p$ -regularity). If  $p$  does not divide  $M$ , then one can take the Frobenius at  $w_0$ , otherwise one can take any non trivial element of the inertia. Since  $\rho(\phi)$  is diagonal in the basis  $\{e_1, e_2\}$  we have:

$$\begin{pmatrix} a_\phi & b_\phi \\ c_\phi & d_\phi \end{pmatrix} = \rho(\phi)^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rho(\phi) = \begin{pmatrix} 0 & b_\phi \\ 0 & 0 \end{pmatrix},$$

where  $b_\phi$  is a root of unity. If  $\varepsilon(\phi) = -1$  then  $a = c = d = 0$ , hence  $\rho$  is reducible as above. Finally, if  $\varepsilon(\phi) = 1$ , then  $b_\phi = b_1 = 1$  which contradicts the regularity of  $\rho$  at  $p$ . Hence  $t_{\mathcal{D}'} = 0$  in all cases, as claimed.

From lemma 2.1 and the exact sequence of  $G_{\mathbb{Q}}$ -modules  $0 \rightarrow \text{ad}^0 \rho \rightarrow \text{ad} \rho \rightarrow \bar{\mathbb{Q}}_p \rightarrow 0$ , one deduces that  $t_{\mathcal{D}} \simeq t_{\mathcal{D}}/t_{\mathcal{D}'}$  is a subspace of the line  $H^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$ , hence its dimension is at most one (we will see in §7 that it is one).  $\square$

## 5. THE ODD DIHEDRAL CASE

In this section we assume that the representation  $\rho$  from §2 is odd and that its projective image  $G$  is dihedral, that is to say isomorphic to  $D_{2n}$  for some  $n \geq 2$ . The study of this case by the same method as in the proof of theorem 4.2 would have required a lengthy examination of the character table of a dihedral group, hence we propose here a slightly different approach.

There exist a quadratic extension  $K$  of  $\mathbb{Q}$  and a character  $\chi : G_K \rightarrow \bar{\mathbb{Q}}_p^\times$  such that

$$(10) \quad \rho \simeq \text{Ind}_K^{\mathbb{Q}} \chi.$$

The extension  $K$  is unique unless the projective image of  $\rho$  is the Klein group  $D_4$ , in which case there are exactly three choices for  $K$  (two imaginary and one real).

Let  $\sigma$  be the non-trivial automorphism of  $K$  and let  $\chi^\sigma$  be the character of  $G_K$  defined by  $\chi^\sigma(g) = \chi(\sigma^{-1}g\sigma)$ . Note that  $\chi$  and  $\chi^\sigma$  play symmetric role, that is  $\chi$  may be replaced by  $\chi^\sigma$  in (10) without change. Since  $\rho$  is irreducible,  $\chi^\sigma \neq \chi$  and  $\ker(\chi^\sigma/\chi)$  defines a cyclic extension  $H$  of  $K$  of degree  $n$ . It is easy to see that  $\rho$  is odd if, and only if,  $H$  is totally complex, which is always the case if  $K$  is imaginary.

One has

$$\mathrm{ad} \rho \simeq 1 \oplus \mathrm{ad}^0 \rho \simeq 1 \oplus \mathrm{ad}^0(\mathrm{Ind}_K^{\mathbb{Q}} \chi) \simeq 1 \oplus \varepsilon_K \oplus \mathrm{Ind}_K^{\mathbb{Q}}(\chi/\chi^\sigma),$$

as  $\bar{\mathbb{Q}}_p[G_{\mathbb{Q}}]$ -modules, where  $\varepsilon_K$  is the quadratic character of  $\mathrm{Gal}(K/\mathbb{Q})$ .

By inflation-restriction, we have

$$H^1(\mathbb{Q}, \mathrm{ad} \rho) = H^1(K, \mathrm{ad} \rho)^{\mathrm{Gal}(K/\mathbb{Q})}.$$

In a basis  $\{e'_1, e'_2\}$  for which  $\rho|_{G_K} = \chi \oplus \chi^\sigma$  one has the following decomposition:

$$(11) \quad \begin{aligned} H^1(K, \mathrm{ad} \rho) &\simeq H^1(K, \chi^\sigma/\chi^\sigma) \oplus H^1(K, \chi/\chi^\sigma) \oplus H^1(K, \chi^\sigma/\chi) \oplus H^1(K, \chi/\chi), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto (a, b, c, d), \end{aligned}$$

where the action of  $\mathrm{Gal}(K/\mathbb{Q})$  exchanges  $a$  and  $d$ , and  $b$  and  $c$ .

If  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^1(\mathbb{Q}, \mathrm{ad} \rho)$ , then  $a + d \in H^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$  and  $a - d \in H^1(\mathbb{Q}, \varepsilon_K)$ .

**Lemma 5.1.** *If  $\rho$  is odd, then  $\dim H^1(K, \chi^\sigma/\chi) = \begin{cases} 1, & \text{if } K \text{ is imaginary,} \\ 2, & \text{if } K \text{ is real.} \end{cases}$*

*Proof.* By inflation-restriction  $H^1(K, \chi^\sigma/\chi) \simeq H^1(H, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi]$  where  $V[\chi^\sigma/\chi]$  denotes the  $\chi^\sigma/\chi$ -isotypic component in the  $\mathrm{Gal}(H/K)$  representation  $V$ .

Let us first assume that  $\chi^\sigma/\chi$  is not quadratic, hence  $\pi = \mathrm{Ind}_K^{\mathbb{Q}}(\chi^\sigma/\chi)$  is irreducible. By lemma 2.3

$$\dim H^1(H, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi] = \dim \mathrm{Hom}_G(\pi, H^1(H, \bar{\mathbb{Q}}_p)) = m_\pi.$$

If  $K$  is imaginary, then  $\det(\pi)(\tau) = \varepsilon_K(\tau)\chi(\tau^2) = (-1) \cdot 1 = -1$ , hence

$$1 = \dim \pi^- \leq m_\pi \leq \dim \pi = 2.$$

Moreover, in this case Leopoldt's conjecture is known for  $H$  by Ax-Baker-Brumer [4], hence  $m_\pi = 1$  as claimed.

If  $K$  is real, the oddness of  $\rho$  implies that  $\tau$  is a non trivial element in  $G$ , whose image in  $\mathrm{Gal}(K/\mathbb{Q})$  is trivial. Hence  $\tau$  is a order 2 central element in  $G$ , that is to say  $H$  is a CM field. It follows that the image of  $\tau$  by any faithful irreducible representation of  $G$  is  $-\mathrm{id}$ , in particular  $2 = \dim \pi^- \leq m_\pi \leq \dim \pi = 2$ . Note that we don't use Leopoldt's conjecture here!

Let us now inspect the case where  $\chi^\sigma/\chi$  is a quadratic character. It follows that  $H$  is a biquadratic extension of  $\mathbb{Q}$  which is a CM field since  $\rho$  is odd. If we denote by  $K'$  and  $K''$  the two other quadratic subfields of  $H$ , it follows that

two among  $K$ ,  $K'$  and  $K''$  are imaginary and one is real. It is easy to see, that there are two characters of  $G_{\mathbb{Q}}$  extending  $\chi^{\sigma}/\chi$ , which are  $\varepsilon_{K'}$  and  $\varepsilon_{K''}$ . Since  $\text{Ind}_K^{\mathbb{Q}}(\chi^{\sigma}/\chi) = \varepsilon_{K'} \oplus \varepsilon_{K''}$ , we have

$$\dim H^1(H, \bar{\mathbb{Q}}_p)[\chi^{\sigma}/\chi] = m_{\varepsilon_{K'}} + m_{\varepsilon_{K''}}.$$

The claim then follows from lemma 2.3, since  $m_{\varepsilon_K} = 1$  for  $K$  imaginary and  $m_{\varepsilon_K} = 0$  for  $K$  real.  $\square$

**5.1. The split case.** Assume that  $p = vv'$  is split in  $K$ . The embedding  $\iota_p$  determines a place  $v$  of  $K$  above  $p$  and allows us to see  $G_{K_v} \simeq G_{\mathbb{Q}_p}$  as a decomposition subgroup of  $G_K$  at  $v$ .

By (10),  $\rho|_{G_K} \simeq \chi \oplus \chi^{\sigma}$ . Establishing the relation between the global characters  $\chi$  and  $\chi^{\sigma}$ , and the local characters  $\psi_1$  and  $\psi_2$  needs a little bit of care. Recall that by  $p$ -regularity  $\psi_1 \neq \psi_2$ . Restricting to  $G_{K_v}$  yields  $\{\chi|_{G_{K_v}}, \chi|_{G_{K_v}}^{\sigma}\} = \{\psi_1, \psi_2\}$  and we use the indeterminacy between  $\chi$  and  $\chi^{\sigma}$  to require that

$$\chi|_{G_{K_v}} = \psi_1, \quad \chi|_{G_{K_v}}^{\sigma} = \psi_2.$$

Note that  $\chi|_{G_{K_{v'}}} = \psi_2$  and  $\chi|_{G_{K_{v'}}}^{\sigma} = \psi_1$ , where  $G_{K_{v'}} = \sigma^{-1}G_{K_v}\sigma$  is a decomposition subgroup of  $G_K$  at  $v'$  (by an abuse of notation  $\sigma$  denotes also an automorphism of  $\bar{\mathbb{Q}}$  acting non-trivially on  $K$ ). Note Since  $G_{\mathbb{Q}_p} \subset G_K$  one can take  $e'_1 = e_1$  and  $e'_2 = e_2$ .

**Proposition 5.2.** *Assume that  $p$  is split in  $K$ . Then  $\dim H^1(K_v, \chi/\chi^{\sigma}) = 1$  and there are natural isomorphisms*

$$\begin{aligned} t_{\mathcal{D}'} &\simeq \ker \left( H^1(K, \chi/\chi^{\sigma}) \rightarrow H^1(K_v, \chi/\chi^{\sigma}) \right), \\ t_{\mathcal{D}} &\simeq t_{\mathcal{D}'} \oplus \ker \left( H^1(K, \bar{\mathbb{Q}}_p) \rightarrow H^1(I_v, \bar{\mathbb{Q}}_p) \right). \end{aligned}$$

*Proof.* Since  $H^0(K_v, \chi/\chi^{\sigma}) = 0$  by  $p$ -regularity and

$$H^2(K_v, \chi/\chi^{\sigma}) \simeq H^0(K_v, \chi^{\sigma}/\chi(1))^{\vee} = 0$$

by Tate's local duality, Tate's local Euler characteristic formula yields

$$\dim H^1(K_v, \chi/\chi^{\sigma}) = [K_v : \mathbb{Q}_p] = 1.$$

By Lemmas 2.1 and 2.2(i) we have

$$t_{\mathcal{D}} = \ker \left( H^1(K, \text{ad } \rho)^{\text{Gal}(K/\mathbb{Q})} \rightarrow (H^1(K_v, \psi_2/\psi_1) \oplus H^1(I_v, \psi_1/\psi_1)) \right).$$

It follows then from (11) that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in t_{\mathcal{D}}$  if, and only if,  $a = d^\sigma$ ,  $b = c^\sigma$ ,

$$c \in \ker \left( H^1(K, \chi/\chi^\sigma) \rightarrow H^1(K_v, \chi/\chi^\sigma) \right), \text{ and}$$

$$d \in \ker \left( H^1(K, \bar{\mathbb{Q}}_p) \rightarrow H^1(I_v, \bar{\mathbb{Q}}_p) \right).$$

To compute  $t_{\mathcal{D}'}$  one should add the extra condition  $a + d = 0$ , that is  $d^\sigma = -d$ . This condition implies that  $d$  belongs to

$$\ker \left( H^1(K, \bar{\mathbb{Q}}_p) \rightarrow H^1(I_v, \bar{\mathbb{Q}}_p) \oplus H^1(I_{v'}, \bar{\mathbb{Q}}_p) \right),$$

which vanishes by lemma 2.2(ii). The claims about  $t_{\mathcal{D}'}$  and  $t_{\mathcal{D}}$  then follow immediately.  $\square$

We now want to compute the dimension of these tangent spaces. If  $K$  is real, then it has no  $\mathbb{Z}_p$ -extensions unramified outside  $v$ , while if  $K$  is imaginary, then it has one such  $\mathbb{Z}_p$ -extension. It follows that

$$(12) \quad \dim t_{\mathcal{D}} = \begin{cases} \dim t_{\mathcal{D}'} & , \text{ if } K \text{ is real,} \\ \dim t_{\mathcal{D}'} + 1 & , \text{ if } K \text{ is imaginary.} \end{cases}$$

The computation of  $\dim t_{\mathcal{D}'}$  is the object of the following theorem.

**Theorem 5.3.** *Assume that  $\rho$  is odd and that  $\rho|_{G_K}$  is reducible for some quadratic field  $K$  in which  $p$  splits. If  $K$  is imaginary, then  $t_{\mathcal{D}'} = 0$  and  $\dim t_{\mathcal{D}} = 1$ . If  $K$  is real, then  $\dim t_{\mathcal{D}'} = \dim t_{\mathcal{D}} = 1$ .*

*Proof.* The embedding  $\iota_p$  determines a place  $w_0$  of  $H$  above  $v$  and allows us to see  $G_{H_{w_0}}$  as a decomposition subgroup of  $G_H$  at  $w_0$ . By Lemma 2.2(i), we have

$$t_{\mathcal{D}'} = \ker \left( H^1(H, \chi/\chi^\sigma)^{\text{Gal}(H/K)} \rightarrow H^1(H_{w_0}, \chi/\chi^\sigma) \right),$$

and since  $\chi/\chi^\sigma$  is trivial on  $G_H$ , we can write

$$t_{\mathcal{D}'} = \ker \left( H^1(H, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi] \rightarrow H^1(H_{w_0}, \bar{\mathbb{Q}}_p) \right).$$

Let  $W$  be such that the following commutative diagram has exact rows:

$$(13) \quad \begin{array}{ccccccc} 0 & \longrightarrow & t_{\mathcal{D}'} & \longrightarrow & H^1(H, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi] & \longrightarrow & H^1(H_{w_0}, \bar{\mathbb{Q}}_p) \\ & & \downarrow & & \downarrow & & \downarrow \simeq \\ 0 & \longrightarrow & W & \longrightarrow & \text{Hom}(\prod_{w|p} \mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi] & \longrightarrow & \text{Hom}(H_{w_0}^\times, \bar{\mathbb{Q}}_p) \end{array}$$

We note that  $\text{Hom}(\prod_{w|p} \mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p) = \text{Hom}(\prod_{w|v} \mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p) \oplus \text{Hom}(\prod_{w|v'} \mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p)$ , each one of the factors being the regular representation of  $\text{Gal}(H/K)$ . Hence the middle term of the second row, given by

$$\text{Hom}(\prod_{w|v} \mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi] \oplus \text{Hom}(\prod_{w|v'} \mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi],$$

has dimension 2, and the last horizontal lower map sends the second factor to 0 (since  $w_0 \nmid v'$ ) but not the first one (since all coordinates of an isotypic vector in the regular representation of an abelian group are no zero). In other words,  $\dim W = 1$ .

To conclude the proof of the theorem, let us first assume that  $K$  is real. By lemma 5.1,  $\dim H^1(H, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi] = 2$ , so the middle vertical map is an isomorphism. It follows that  $t_{\mathcal{D}'} = W$  has dimension 1 in this case and by (12) the space  $t_{\mathcal{D}}$  has dimension 1 too.

Finally, if  $K$  is imaginary, by lemma 5.1,  $H^1(H, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi]$  is a line. Moreover, it has a no zero image in  $\text{Hom}(\prod_{w|v} \mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p)[\chi^\sigma/\chi]$ . In fact, by Minkowski's proof of Dirichlet's unit theorem, any element of  $\text{Hom}(\prod_{w|v} \mathcal{O}_{H,w}^\times, \bar{\mathbb{Q}}_p)$  which is trivial on global units has to factor through the norm. Hence it does not belong to the  $\chi^\sigma/\chi$ -isotypic component and the previous analysis then shows that  $t_{\mathcal{D}'} = 0$ .  $\square$

**5.2. The inert and ramified cases.** Suppose that  $p$  is inert or ramified in  $K$ , and let  $v$  be the unique place of  $K$  above  $p$ . By ordinarity  $\chi$  is unramified at  $v$ , hence  $\chi|_{G_{K_v}} = \chi|_{G_{K_v}}^\sigma$ . It follows that  $v$  splits completely in  $H$  and that  $\psi_2\psi_1^{-1}$  is the quadratic character of  $\text{Gal}(K_v/\mathbb{Q}_p)$ , in particular  $\rho$  is necessarily  $p$ -regular.

Let  $\{e'_1, e'_2\}$  be a basis in which  $\rho|_{G_K} = \chi \oplus \chi^\sigma$ . If  $p$  is inert in  $K$ , then by rescaling this basis one can assume that  $\rho(\text{Frob}_v) = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ , in which case  $\rho(\text{Frob}_v)$  is diagonal in the basis  $\{e'_1 + e'_2, e'_1 - e'_2\}$ . Similarly, if  $p$  is ramified in  $K$ , then one can assume that  $\rho|_{I_v} = 1 \oplus \varepsilon_K$  in the basis  $\{e'_1 + e'_2, e'_1 - e'_2\}$ , so in all cases one can assume that  $e'_1 + e'_2 = e_1$  and  $e'_1 - e'_2 = e_2$ .

With the same notations as in (11), a simple matrix computation shows that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in t_{\mathcal{D}}$  if, and only if:

$$a = d^\sigma, b = c^\sigma, c_v + d_v = c_v^\sigma + d_v^\sigma \text{ and } d_v + d_v^\sigma - c_v - c_v^\sigma \text{ is unramified,}$$

the last two conditions taking place in  $H^1(K_v, \bar{\mathbb{Q}}_p)$ . Here  $c_v = i(c)$  and  $d_v = j(d)$ , where  $i$  and  $j$  are the following two restriction homomorphisms:

$$(14) \quad c \in H^1(K, \chi/\chi^\sigma) \xrightarrow{i} \text{im} (H^1(K_v, \bar{\mathbb{Q}}_p) \rightarrow H^1(I_v, \bar{\mathbb{Q}}_p)) \xleftarrow{j} H^1(K, \bar{\mathbb{Q}}_p) \ni d.$$

To compute  $t_{\mathcal{D}'}$  one should add the extra condition  $a + d = 0$ , that is  $d^\sigma = -d$ . If  $H$  is biquadratic, then by ordinarity  $p$  splits in one of its quadratic subfields and theorem 5.3 applies. Otherwise  $K$  is uniquely defined and we have:

**Theorem 5.4.** *Assume that  $\rho|_{G_K}$  is reducible for a unique quadratic field  $K$ . If  $\rho$  is odd and if  $p$  is inert or ramified in  $K$ , then  $\dim t_{\mathcal{D}} = 1$  and  $t_{\mathcal{D}'} = 0$ .*

*Proof.* Note that  $\text{im}(\text{H}^1(K_v, \bar{\mathbb{Q}}_p) \rightarrow \text{H}^1(I_v, \bar{\mathbb{Q}}_p)) \simeq \text{Hom}(\mathcal{O}_{K,v}^\times, \bar{\mathbb{Q}}_p)$  has dimension 2. Assume first that  $K$  is real. Then  $d^\sigma - d \in \text{H}^1(\mathbb{Q}, \varepsilon_K) = 0$ , hence  $d = d^\sigma \in \text{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$  which is a line. Moreover by lemma 5.1,  $\dim \text{H}^1(K, \chi/\chi^\sigma) = 2$ . It follows from here and from lemma 2.2(ii) that the homomorphism  $j$  in (14) is injective while  $i$  is an isomorphism. Hence

$$t_{\mathcal{D}} = \left\{ \begin{pmatrix} d & \sigma \circ i^{-1} \circ j(d) \\ i^{-1} \circ j(d) & d \end{pmatrix} \mid d \in \text{H}^1(K, \bar{\mathbb{Q}}_p) \right\}$$

is a line and if we add the condition that  $d + d^\sigma = 0$  we obtain  $t_{\mathcal{D}'} = 0$ .

Assume next that  $K$  is imaginary. By lemma 5.1,  $\dim \text{H}^1(K, \chi/\chi^\sigma) = 1$ . Moreover  $\dim \text{H}^1(K, \bar{\mathbb{Q}}_p) = 2$ , since  $\dim \text{H}^1(\mathbb{Q}, \varepsilon_K) = \dim \text{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p) = 1$ . It follows from here and from lemma 2.2(ii) that the homomorphism  $i$  in (14) is injective while  $j$  is an isomorphism. Hence

$$t_{\mathcal{D}} = \left\{ \begin{pmatrix} j^{-1} \circ i(c) & \sigma(c) \\ c & \sigma \circ j^{-1} \circ i(c) \end{pmatrix} \mid c \in \text{H}^1(K, \chi/\chi^\sigma) \right\}$$

is a line. It remains to show that  $\dim t_{\mathcal{D}'} = 0$ . One has:

$$(15) \quad t_{\mathcal{D}'} \simeq \{c \in \text{H}^1(K, \chi/\chi^\sigma) \mid \sigma \circ i(c) = -i(c)\},$$

where by an abuse of notation we denote by  $\sigma$  also the automorphism of  $\text{Hom}(\mathcal{O}_{K,v}^\times, \bar{\mathbb{Q}}_p)$ .

Suppose that  $t_{\mathcal{D}'}$  has a non-zero element and denote by

$$\varphi : \prod_{w|p} \mathcal{O}_{H,w}^\times \mapsto \bar{\mathbb{Q}}_p$$

the corresponding character in (13). By (15), up to a non-zero scalar, the restriction of  $\varphi$  to a sufficiently small finite index subgroup of  $(\mathcal{O}_H \otimes \mathbb{Z}_p)^\times$  is given by:

$$\varphi((u_w)_{w|p}) = \prod_{i=0}^{n-1} \zeta^{-i} (\log_p(u_{g^i(w_0)}) - \log_p(\sigma u_{g^i(w_0)})),$$

where  $g$  is a generator of  $\text{Gal}(H/K)$ . Let us examine whether such a  $\varphi$  could be algebraic, that is to say trivial on the global units. Since  $\sigma$  lifts to a complex conjugation of  $H$ , by Minkowski's proof of Dirichlet's unit theorem, there exists

$u_0 \in \mathcal{O}_H^\times$  such that  $u_0, g(u_0), \dots, g^{n-2}(u_0)$  is a basis of  $\mathcal{O}_H^\times \otimes \mathbb{Q}$ . Since  $K$  is imaginary, we have:

$$u_0 \cdot g(u_0) \cdot \dots \cdot g^{n-1}(u_0) = 1.$$

Write  $\sigma(u_0) = \prod_{i=0}^{n-1} g^i(u_0)^{a_i}$  with  $\sum_{i=0}^{n-1} a_i = 0$  (barycentric coordinates). Since  $G$  is dihedral  $\sigma g^i = g^{-i} \sigma$ , hence

$$\sigma g^i(u_0) = \prod_{j=0}^{n-1} g^j(u_0)^{a_{i+j}}.$$

We deduce that  $\varphi(u_0) = 1$  if, and only if, one has the following equality in  $\mathcal{O}_H^\times \otimes \bar{\mathbb{Q}}$ :

$$\prod_{i=0}^{n-1} g^i(u_0)^{\zeta^i} = \prod_{i=0}^{n-1} \sigma g^i(u_0)^{\zeta^i} = \prod_{i=0}^{n-1} g^i(u_0)^{\sum_{j=0}^{n-1} a_{i+j} \zeta^j}.$$

By theorem 3.1 and by the uniqueness of barycentric coordinates  $\zeta^i = \sum_{j=0}^{n-1} a_j \zeta^{j-i}$ , hence  $\zeta^{2i} = \sum_{j=0}^{n-1} a_j \zeta^j$  is independent of  $i$  and therefore  $\zeta^2 = 1$ . This implies that  $H$  is a biquadratic, which is a contradiction.  $\square$

## 6. THE EIGENCURVE

As in the introduction, we let  $\mathcal{C}$  be the  $p$ -adic Eigencurve of tame level  $N$  constructed using the Hecke operators  $U_p$  and  $T_\ell$   $\ell \nmid Np$ . It is reduced and equipped with a flat and locally finite morphism  $\kappa : \mathcal{C} \rightarrow \mathcal{W}$ , called the weight map (we refer the reader to [9], in the case  $N = 1$ , and to [5] for the general case). By construction, there exist analytic functions  $U_p$  and  $T_\ell$  ( $\ell \nmid N$ ) belonging to  $\mathcal{O}(\mathcal{C})$  (and even to  $\mathcal{O}(\mathcal{C})^\times$  for  $U_p$ ). The locus where  $|U_p| = 1$  is open and closed in  $\mathcal{C}$ , and is called the ordinary part of the Eigencurve; it is closely related to Hida families. There exists (see [6, §7]) a continuous pseudo-character

$$(16) \quad G_{\mathbb{Q}, Np} \rightarrow \mathcal{O}(\mathcal{C})$$

sending  $\text{Frob}_\ell$  to  $T_\ell$  for all  $\ell \nmid Np$ .

If  $f_\alpha$  is as in the introduction, its system of eigenvalues corresponds to a point  $x \in \mathcal{C}(\bar{\mathbb{Q}}_p)$  such that  $\kappa(x)$  has finite order. Since  $U_p(x) = \alpha$  is a  $p$ -adic unit,  $x$  actually lies on the ordinary part of  $\mathcal{C}$ . Denote by  $\mathcal{T}$  the completed local ring of  $\mathcal{C}$  at  $x$  (maximal ideal  $\mathfrak{m}$ ).

**Proposition 6.1.** *There exists a continuous representation*

$$\rho_{\mathcal{T}} : G_{\mathbb{Q}, Np} \rightarrow \text{GL}_2(\mathcal{T}),$$



such that  $\text{Tr } \rho_{\mathcal{T}}(\text{Frob}_{\ell}) = T_{\ell}$  for all  $\ell \nmid Np$ . The reduction of  $\rho_{\mathcal{T}}$  modulo  $\mathfrak{m}$  is isomorphic to  $\rho_f$ . If  $f$  is regular at  $p$ , then  $\rho_{\mathcal{T}}$  is ordinary at  $p$  in the sense that:

$$(17) \quad (\rho_{\mathcal{T}})|_{G_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \psi_{2,\mathcal{T}} & * \\ 0 & \psi_{1,\mathcal{T}} \end{pmatrix},$$

where  $\psi_{1,\mathcal{T}} : G_{\mathbb{Q}_p} \rightarrow \mathcal{T}^{\times}$  is the unramified character sending  $\text{Frob}_p$  to  $U_p$ .

*Proof.* Consider the pseudo-character  $G_{\mathbb{Q},Np} \rightarrow \mathcal{T}$  obtained by composing (16) with the algebra homomorphism  $\mathcal{O}(\mathbb{C}) \rightarrow \mathcal{T}$ . It is a two dimensional pseudo-character taking values in a strictly henselian (even complete) local ring, whose residual pseudo-character is given by the trace of the irreducible representation  $\rho_f$ . By a theorem of Nyssen [17] and Rouquier [18] there exists a continuous representation  $\rho_{\mathcal{T}}$  as in the statement. It remains to show that  $\rho_{\mathcal{T}}$  is ordinary at  $p$ .

Denote by  $\mathcal{M}$  be the free rank two  $\mathcal{T}$ -module on which  $\rho_{\mathcal{T}}$  acts. Let  $(K_i)_{1 \leq i \leq r}$  be the fields obtained by localizing  $\mathcal{T}$  at its minimal primes and put  $V_i = \mathcal{M} \otimes_{\mathcal{T}} K_i$  ( $1 \leq i \leq r$ ). The  $K_i[G_{\mathbb{Q},Np}]$ -module  $V_i$  is nothing else but the Galois representation attached to a Hida family specializing to  $f_{\alpha}$ . By [19] there exists a short exact sequence of  $K_i[G_{\mathbb{Q}_p}]$ -modules:

$$0 \rightarrow V_i^+ \rightarrow V_i \rightarrow V_i^- \rightarrow 0,$$

where  $V_i^-$  is a line on which  $G_{\mathbb{Q}_p}$  acts via the unramified character sending  $\text{Frob}_p$  to  $U_p$ .

Since  $\mathcal{T}$  is reduced, the natural homomorphism  $\mathcal{T} \rightarrow \prod_i K_i$  is injective. Since  $\mathcal{M}$  is free over  $\mathcal{T}$ , so is the natural homomorphism  $\mathcal{M} \rightarrow \prod_i V_i$ . This leads to a short exact sequence of  $\mathcal{T}[G_{\mathbb{Q}_p}]$ -modules:

$$(18) \quad 0 \rightarrow \mathcal{M}^+ \rightarrow \mathcal{M} \rightarrow \mathcal{M}^- \rightarrow 0$$

where  $\mathcal{M}^+ = \mathcal{M} \cap \prod_i V_i^+$  and  $\mathcal{M}^-$  is the image of  $\mathcal{M} \subset \prod_i V_i$  in  $\prod_i V_i^-$ .

One deduces from here an exact sequence of  $\bar{\mathbb{Q}}_p[G_{\mathbb{Q}_p}]$ -modules:

$$(19) \quad \mathcal{M}^+/\mathfrak{m}\mathcal{M}^+ \rightarrow \mathcal{M}/\mathfrak{m}\mathcal{M} \rightarrow \mathcal{M}^-/\mathfrak{m}\mathcal{M}^- \rightarrow 0$$

where  $G_{\mathbb{Q}_p}$  acts on  $\mathcal{M}^-/\mathfrak{m}\mathcal{M}^-$  via  $\psi_1$  and on  $\mathcal{M}^+/\mathfrak{m}\mathcal{M}^+$  via  $\psi_2$ . Since  $\mathcal{M}/\mathfrak{m}\mathcal{M} \simeq \psi_1 \oplus \psi_2$  as  $\bar{\mathbb{Q}}_p[G_{\mathbb{Q}_p}]$ -modules and since  $\psi_1 \neq \psi_2$  by assumption, it follows that  $\dim_{\bar{\mathbb{Q}}_p}(\mathcal{M}^-/\mathfrak{m}\mathcal{M}^-) = 1$ , hence, by Nakayama's lemma,  $\mathcal{M}^-$  is free of rank one over  $\mathcal{T}$ . It follows that (18) splits, hence the first homomorphism in (19) is injective and  $\mathcal{M}^+$  is free of rank one over  $\mathcal{T}$  by the same argument as for  $\mathcal{M}^-$ .  $\square$

Let  $\Lambda$  be the completed local ring of  $\mathcal{W}$  at  $\kappa(x)$  (maximal ideal  $\mathfrak{m}_\Lambda$ ). It is isomorphic to a power series ring in one variable over  $\bar{\mathbb{Q}}_p$ . The weight map  $\kappa$  induces a finite and flat homomorphism  $\Lambda \rightarrow \mathcal{T}$  of local reduced complete rings. The algebra of the fiber of  $\kappa$  at  $x$  is a local Artinian  $\bar{\mathbb{Q}}_p$ -algebra given by

$$(20) \quad \mathcal{T}' := \mathcal{T}/\mathfrak{m}_\Lambda \mathcal{T}$$

The dimension of  $\mathcal{T}'$  over  $\bar{\mathbb{Q}}_p$  is the degree of  $\kappa$  at  $x$ , which is one if, and only if,  $\kappa$  is etale at  $x$ .

## 7. PROOF OF THE MAIN THEOREM AND ITS COROLLARY

Consider the deformation functors  $\mathcal{D}$  and  $\mathcal{D}'$  from §2 associated to  $\rho = \rho_f$  and the unramified character  $\psi_1$  of  $G_{\mathbb{Q}_p}$  sending  $\text{Frob}_p$  to  $\alpha$ . The functor  $\mathcal{D}$  is pro-representable by a local Noetherian  $\bar{\mathbb{Q}}_p$ -algebra  $\mathcal{R}$ , while  $\mathcal{D}'$  is representable by a local Artinian ring  $\mathcal{R}'$  with residue field  $\bar{\mathbb{Q}}_p$ . As usual, one has isomorphisms of  $\bar{\mathbb{Q}}_p$ -vector spaces

$$t_{\mathcal{D}} \cong \mathfrak{m}_{\mathcal{R}}/\mathfrak{m}_{\mathcal{R}}^2 \text{ and } t_{\mathcal{D}'} \cong \mathfrak{m}_{\mathcal{R}'}/\mathfrak{m}_{\mathcal{R}'}^2.$$

By proposition 6.1, the representation  $\rho_{\mathcal{T}}$  yields a continuous homomorphism of local complete Noetherian  $\bar{\mathbb{Q}}_p$ -algebras

$$(21) \quad \mathcal{R} \rightarrow \mathcal{T}.$$

Let  $A$  be any local Artinian ring with maximal ideal  $\mathfrak{m}_A$  and residue field  $A/\mathfrak{m}_A = \bar{\mathbb{Q}}_p$ . A deformation of  $\det(\rho_f)$  to  $A^\times$  is equivalent to a continuous homomorphism from  $G_{\mathbb{Q}, Np}$  to  $1 + \mathfrak{m}_A$ . By class field theory (and since  $1 + \mathfrak{m}_A$  does not contain elements of finite order), the latter are elements of:

$$\text{Hom}((\mathbb{Z}/N)^\times \times \mathbb{Z}_p^\times, 1 + \mathfrak{m}_A) = \text{Hom}(1 + p\mathbb{Z}_p, 1 + \mathfrak{m}_A).$$

It follows that the universal deformation ring of  $\det(\rho_f)$  is given by  $\bar{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]]$  which endows  $\mathcal{R}$  with a structure of  $\bar{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]]$ -algebra. By definition  $\det(\rho_{\mathcal{R}})$  factors through  $\bar{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]]^\times \rightarrow \mathcal{R}^\times$  and

$$\mathcal{R}' = \mathcal{R}/\mathfrak{m}_{\bar{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]]} \mathcal{R}.$$

Since the Langlands correspondence relates the determinant to the central character, the homomorphism (21) is  $\Lambda$ -linear, in the sense that the following diagram

commutes:

$$\begin{array}{ccc} \bar{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]] & \longrightarrow & \mathcal{R} \\ \downarrow & & \downarrow \\ \Lambda & \longrightarrow & \mathcal{T} \end{array}$$

**Lemma 7.1.** *The natural homomorphism  $\bar{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]] \rightarrow \Lambda$  is an isomorphism and one has a natural homomorphism of local Artinian  $\bar{\mathbb{Q}}_p$ -algebras:*

$$(22) \quad \mathcal{R}' \rightarrow \mathcal{T}'.$$

*In particular, the homomorphism  $\det(\rho_{\mathcal{T}}) : G_{\mathbb{Q}, Np} \rightarrow \mathcal{T}^{\times}$  factors through  $\Lambda^{\times} \rightarrow \mathcal{T}^{\times}$  and  $\mathcal{T}'$  is the largest quotient of  $\mathcal{T}$  to which  $\rho_f$  can be deformed with constant determinant.*

*Proof.* Since  $\Lambda$  is equidimensional of dimension 1 and  $\bar{\mathbb{Q}}_p[[1 + p\mathbb{Z}_p]] \rightarrow \Lambda$  is surjective, it is an isomorphism of regular local rings of dimension 1. The homomorphism (22) is obtained by reducing (21) modulo  $\mathfrak{m}_{\Lambda}$ .  $\square$

**Proposition 7.2.** *The homomorphisms (21) and (22) are surjective.*

*Proof.* Since  $\mathcal{T}$  is topologically generated over  $\Lambda$  by  $U_p$  and  $T_{\ell}$  for  $\ell \nmid Np$  it suffices to check that those elements are in the image of  $\mathcal{R}$ . For  $\ell \nmid Np$ ,  $T_{\ell} = \text{Tr } \rho_{\mathcal{T}}(\text{Frob}_{\ell})$  is the image of the trace of  $\rho_{\mathcal{R}}(\text{Frob}_{\ell})$ .

Recall that the restriction of the universal deformation  $\rho_{\mathcal{R}} : G_{\mathbb{Q}, Np} \rightarrow \text{GL}_2(\mathcal{R})$  to  $G_{\mathbb{Q}_p}$  is reducible, extension of an unramified character  $\psi_{1, \mathcal{R}}$  lifting  $\psi_1$  by a character  $\psi_{2, \mathcal{R}}$ . By proposition 6.1,  $U_p$  is the image of  $\psi_{1, \mathcal{R}}(\text{Frob}_p)$ .  $\square$

**Theorem 7.3.** *The homomorphisms (21) and (22) are isomorphisms and  $\mathcal{T}$  is regular.*

*Proof.* It follows from theorems 4.2, 5.3 and 5.4 that the tangent space  $t_{\mathcal{D}}$  of  $\mathcal{R}$  has dimension 1. Since  $\mathcal{T}$  is equidimensional of dimension 1, proposition 7.2 implies that (21) is an isomorphism of regular local rings of dimension 1. The fact that (22) is an isomorphism then easily follows from lemma 7.1.  $\square$

We have already proved the first part of theorem 1.1: the Eigencurve is smooth at  $f_{\alpha}$ . Moreover, it is étale at  $f_{\alpha}$  over the weight space if and only if  $\mathcal{T}'$  is of dimension 1, hence if and only if  $\mathcal{R}'$  is of dimension 1, which by theorems 4.2, 5.3

and 5.4 is equivalent to  $f$  not having multiplication by a real quadratic field in which  $p$  splits. This completes the proof of theorem 1.1.

Assume that  $f$  has CM by a field  $K$  in which  $p$  splits. For the corollary we need to prove that any irreducible component of  $\mathcal{C}$  containing  $f_\alpha$  also has CM by  $K$ . Let  $\mathcal{C}_K$  be the Eigencurve constructed using Buzzard's eigenvariety machine [5] using the submodule of overconvergent modular forms with CM by  $K$ . By the eigenvariety machine,  $\mathcal{C}_K$  is equidimensional of dimension one, and there is a natural closed immersion from  $\mathcal{C}_K$  to  $\mathcal{C}$  by a simpler analogue of the main result of [7]. Since  $f_\alpha$  belongs to the image of  $\mathcal{C}_K$  and since  $\mathcal{C}$  has a unique irreducible component containing  $f_\alpha$ , it follows that this component is the image of  $\mathcal{C}_K$ , hence the corollary.

## 8. THE FULL EIGENCURVE

Let  $\mathcal{C}^{\text{full}}$  be the  $p$ -adic Eigencurve of tame level  $N$  constructed using the Hecke operators  $T_\ell$  for  $\ell \nmid Np$  and  $U_\ell$  for  $\ell \mid Np$ . It comes with a locally finite surjective morphism  $\mathcal{C}^{\text{full}} \rightarrow \mathcal{C}$  compatible with all other structures (which is not an isomorphism when  $N > 1$ ), yielding by composition a two dimensional pseudo-character  $G_{\mathbb{Q}, Np} \rightarrow \mathcal{O}(\mathcal{C}^{\text{full}})$ . There is a natural bijection between  $\mathcal{C}^{\text{full}}(\bar{\mathbb{Q}}_p)$  and the set of systems of eigenvalues of overconvergent eigenforms with finite slope of tame level dividing  $N$  and weight in  $\mathcal{W}(\bar{\mathbb{Q}}_p)$ , sending  $x$  to the system of eigenvalues  $T_\ell(x)$  ( $\ell \nmid Np$ ) and  $U_\ell(x)$  ( $\ell \mid Np$ ) appearing in the space of overconvergent modular forms of weight  $\kappa(x)$ .

Keep the notations from §6 and let  $\mathcal{T}^{\text{full}}$  be the completed local ring of  $\mathcal{C}^{\text{full}}$  at the point  $x$  defined by  $f_\alpha$  (maximal ideal  $\mathfrak{m}^{\text{full}}$ ) and  $\mathcal{T}'^{\text{full}} := \mathcal{T}^{\text{full}}/\mathfrak{m}_\Lambda \mathcal{T}^{\text{full}}$ . The morphism  $\mathcal{C}^{\text{full}} \rightarrow \mathcal{C}$  yields a homomorphism  $\mathcal{T} \rightarrow \mathcal{T}^{\text{full}}$  and by proposition 6.1 there exists a continuous representation

$$\rho_{\mathcal{T}}^{\text{full}} : G_{\mathbb{Q}, Np} \rightarrow \text{GL}_2(\mathcal{T}^{\text{full}}),$$

such that  $\text{Tr } \rho_{\mathcal{T}}^{\text{full}}(\text{Frob}_\ell) = T_\ell$ , for all  $\ell \nmid Np$ . Moreover  $\rho_{\mathcal{T}}^{\text{full}}$  is ordinary at  $p$  and its reduction modulo  $\mathfrak{m}^{\text{full}}$  is isomorphic to  $\rho_f$ .

Denote by  $\mathcal{M}$  be the free rank two  $\mathcal{T}^{\text{full}}$ -module on which  $\rho_{\mathcal{T}}^{\text{full}}$  acts.

**Proposition 8.1.** *Let  $\ell$  be a prime dividing  $N$ . Then  $\rho_{\mathcal{T}}^{\text{full}}(I_\ell)$  is finite. Moreover:*

- (i) *If  $a_\ell \neq 0$ , then  $\mathcal{M}^{I_\ell}$  is a free rank one direct summand of  $\mathcal{M}$  on which  $\rho_{\mathcal{T}}^{\text{full}}(\text{Frob}_\ell)$  acts as multiplication by  $U_\ell$ .*

(ii) If  $a_\ell = 0$ , then  $\mathcal{M}^{I_\ell} = 0$  and  $U_\ell = 0$  in  $\mathcal{T}^{\text{full}}$ .

*Proof.* To prove that  $\rho_{\mathcal{T}}^{\text{full}}(I_\ell)$  is finite, it is enough to prove that  $\rho_{\mathcal{T}}(I_\ell)$  is finite since  $\rho_{\mathcal{T}}^{\text{full}}$  factors, by construction, through  $\rho_{\mathcal{T}}$ . Fix a non-zero continuous homomorphism  $t_p : I_\ell \rightarrow \mathbb{Z}_p$ . By Grothendieck's monodromy theorem in family (see [1, Lemma 7.8.14]), there exists a unique nilpotent matrix  $\mathcal{N} \in M_2(\mathcal{T})$  such that  $\rho(g) = \exp(t_p(g)\mathcal{N})$  on an open subgroup of  $I_\ell$ . Thus it suffices to show that  $\mathcal{N} = 0$ . Let  $s$  be a generic point of  $\text{Spec}(\mathcal{T})$ ,  $\rho_s$  the attached representation and  $\mathcal{N}_s$  its monodromy operator. If  $\mathcal{N}_s \neq 0$ , then as is well known  $\rho_s|_{G_\ell}$  is an extension of 1 by the  $p$ -adic cyclotomic character, hence the same is true for  $\rho_{\mathcal{T}} \bmod \mathfrak{m} \simeq \rho_f$ , which is impossible since  $\rho_f$  has finite image and  $\ell$  is not a root of unity. Thus  $\mathcal{N}_s = 0$  for all generic points  $s$ . Since  $\mathcal{T}$  is reduced, this easily implies that  $\mathcal{N} = 0$ .

Hence  $\rho_{\mathcal{T}}(I_\ell)$  is finite. It follows that the submodule  $\mathcal{M}^{I_\ell}$  is a direct summand of  $\mathcal{M}$  (as it is the image of the projector  $P = \frac{1}{|\rho_{\mathcal{T}}^{\text{full}}(I_\ell)|} \sum_{g \in \rho_{\mathcal{T}}(I_\ell)} g$ ), and that the natural homomorphism  $\mathcal{M}^{I_\ell} / \mathfrak{m}^{\text{full}} \mathcal{M}^{I_\ell} \rightarrow (\mathcal{M} / \mathfrak{m}^{\text{full}} \mathcal{M})^{I_\ell}$  is an isomorphism (it is injective since  $\mathcal{M}^{I_\ell}$  is direct summand and surjective, because  $x \in (\mathcal{M} / \mathfrak{m}^{\text{full}} \mathcal{M})^{I_\ell}$  is the image of  $P(y)$ , where  $y$  any lift of  $x$  in  $\mathcal{M}$ ). The space  $(\mathcal{M} / \mathfrak{m}^{\text{full}} \mathcal{M})^{I_\ell} = \rho_f^{I_\ell}$  has dimension 1 if  $a_\ell \neq 0$ , and 0 if  $a_\ell = 0$ . By Nakayama,  $\mathcal{M}^{I_\ell}$  is 0 if  $a_\ell = 0$ , and is free of rank one if  $a_\ell \neq 0$ .

For the assertions concerning  $U_\ell$ , choose an affinoid neighborhood  $U$  of  $x$  in  $\mathcal{C}^{\text{full}}$  such that there exists a representation  $\rho_U : G_{\mathbb{Q}, Np} \rightarrow \text{GL}(\mathcal{M}_U)$ , where  $\mathcal{M}_U$  is a free module over  $\mathcal{O}(U)$  of rank 2, such that  $\text{Tr } \rho_U : G_{\mathbb{Q}, Np} \rightarrow \mathcal{O}(U)$  is the natural pseudo-character (in particular  $\mathcal{M}_U \otimes_{\mathcal{O}} \mathcal{T}^{\text{full}} \simeq \mathcal{M}$  as  $G_{\mathbb{Q}, Np}$ -modules). By standard arguments, there exists a Zariski-dense set of classical points  $y \in U$  such that  $(\mathcal{M}_{U,y})^{I_\ell} = (\mathcal{M}_U^{I_\ell})_y$ , where the subscript  $y$  means the fiber at  $y$ . If  $\mathcal{M}^{I_\ell}$  has rank one (resp. zero), so has  $\mathcal{M}_U^{I_\ell}$  up to shrinking  $U$ , and so has  $(\mathcal{M}_{U,y})^{I_\ell}$  for  $y$  as above, meaning that the action of  $\text{Frob}_\ell$  on  $(\mathcal{M}_{U,y})^{I_\ell}$  is the multiplication by  $U_\ell(y)$  (resp. that  $U_\ell(y) = 0$ ). By Zariski density of those  $y$ 's, this means that  $\rho_U(\text{Frob}_\ell)$  acts by multiplication by  $U_\ell$  on  $\mathcal{M}_U$  (resp. that  $U_\ell = 0$  in  $\mathcal{O}(U)$ ), which implies the desired results.  $\square$

Let us also introduce a deformation problem  $\mathcal{D}^{\text{full}}$  where

$$\mathcal{D}^{\text{full}}(A) = \{\rho_A \in \mathcal{D}(A), \rho_A^{I_\ell} \text{ is a free of rank one over } A, \text{ for all } \ell \mid N, a_\ell \neq 0\}.$$

It is clear that  $\mathcal{D}^{\text{full}}$  is a representable, by a complete local ring  $\mathcal{R}^{\text{full}}$  and that there is a natural surjective local homomorphism  $\mathcal{R} \twoheadrightarrow \mathcal{R}^{\text{full}}$ .

By proposition 8.1 the representation  $\rho_{\mathcal{T}}^{\text{full}}$  defines a point of  $\mathcal{D}^{\text{full}}$ , hence a homomorphism of local Artinian  $\bar{\mathbb{Q}}_p$ -algebras

$$\pi : \mathcal{R}^{\text{full}} \rightarrow \mathcal{T}^{\text{full}}.$$

**Theorem 8.2.** *The homomorphism  $\mathcal{R}^{\text{full}} \rightarrow \mathcal{T}^{\text{full}}$  is surjective.*

*Proof.* By proposition 7.2 it is enough to prove that  $U_\ell$  belongs to the image for  $\ell \mid N$ . If  $a_\ell = 0$ , then by proposition 8.1(ii)  $U_\ell = 0$  in  $\mathcal{T}^{\text{full}}$  and there is nothing to prove. If  $a_\ell \neq 0$ , then by proposition 8.1(i)  $U_\ell$  is the image of the element of  $\mathcal{R}^{\text{full}}$  by which  $\text{Frob}_\ell$  acts on the free rank one module  $\rho_{\mathcal{R}^{\text{full}}}^{I_\ell}$ .  $\square$

**Corollary 8.3.** *Suppose that  $f$  is regular at  $p$ . Then  $\mathcal{C}^{\text{full}}$  is smooth at  $f_\alpha$  and  $\mathcal{C}^{\text{full}} \rightarrow \mathcal{C}$  is an isomorphism locally at  $f_\alpha$ .*

*Proof.* The composition  $\mathcal{R} \twoheadrightarrow \mathcal{R}^{\text{full}} \twoheadrightarrow \mathcal{T}^{\text{full}}$  is surjective as a composition of two surjective maps, and factors through  $\mathcal{T}$ , yielding a surjective homomorphism  $\mathcal{T} \twoheadrightarrow \mathcal{T}^{\text{full}}$ . Since  $\mathcal{T} \rightarrow \mathcal{T}^{\text{full}}$  is injective by definition, it is an isomorphism. Equivalently, since  $\mathcal{T}^{\text{full}}$  is equidimensional of dimension one, and  $\mathcal{T}$  is regular by theorem 7.3, one has  $\mathcal{T} \simeq \mathcal{T}^{\text{full}}$  and  $\mathcal{R}^{\text{full}} \simeq \mathcal{T}^{\text{full}}$ .  $\square$

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